ISRAEL JOURNAL OF MATHEMATICS **141** (2004), 211–220

ON TATE–SHAFAREVICH GROUPS OVER GALOIS EXTENSIONS

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ABSTRACT

Let A be an abelian variety defined over a number field K. Let L be a finite Galois extension of K with Galois group G and let $Ш(A/K)$ and $Ш(A/L)$ denote, respectively, the Tate–Shafarevich groups of A over K and of A over L. Assuming these groups are finite, we compute $[Ш(A/L)^G]/[Ш(A/K)]$ and $[Ш(A/K)]/[N(Ш(A/L))]$, where $[X]$ is the order of a finite abelian group X. Especially, when L is a quadratic extension of K, we derive a simple formula relating $[Ш(A/L)]$, $[Ш(A/K)]$, and $[Ш(A^{\chi}/K)]$ where A^{χ} is the twist of A by the non-trivial character χ of G.

1. Introduction

Let L/K be a finite Galois extension of number fields with Galois group G. Write $\overline{K}$, G_K, M_K, K_v for the algebraic closure of K, $\mathrm{Gal}(\overline{K}/K)$, a complete set of places on K, the completion of K at the place $v \in M_K$, respectively. Fix a place $v_L \in M_L$ lying above v for each $v \in M_K$. Denote $\mathrm{Gal}(L_w/K_w)$ by G_w for $w \in M_L$.

Let A be an abelian variety defined over K. The conjecture of Birch and Swinnerton-Dyer predicts the leading coefficient of the Taylor expansion for the L-function $L(A/K, s)$ attached to A/K at $s = 1$. Denote by $BSD(A/K)$ the conjectured leading coefficient, which is defined by the product of several algebraic invariants including the order of the Tate–Shafarevich group (see [7] or [18]). The constant $BSD(A/K)$ is an isogeny invariant (see [18, Theorem 2.1]).

Received April 15, 2002 and in revised form May 19, 2003

Assume L is a quadratic extension and A^χ denotes the quadratic twist by the non-trivial character χ of G. Milne [7] showed that if the Tate–Shafarevich groups are finite,

$$BSD(A/L) = BSD(A/K) \cdot BSD(A^\chi/K).$$

Let $Ш(A/K)$ and $Ш(A/L)$ denote the Tate–Shafarevich groups of A over K and of A over L, respectively. We assume throughout that these groups are finite. We write $[X]$ for the order of a finite abelian group X. Note that the Tate–Shafarevich group is not an isogeny invariant and, in general,

$$[Ш(A/L)] \neq [Ш(A/K)][Ш(A^\chi/K)].$$

On the difference there are partial results in [5, Corollary 4.6], [7, Corollary to Theorem 3] and [9, Theorem 4.8]. In this paper we derive a simple formula relating the orders of $Ш(A/L)$, $Ш(A/K)$ and $Ш(A^\chi/K)$.

Main Theorem: *Assume that the Tate–Shafarevich groups are finite. Let A' be the dual variety of A. Then*

$$\frac{[Ш(A/K)][Ш(A^\chi/K)]}{[Ш(A/L)]} = \frac{[\widehat{H}^0(G, A'(L))][H^1(G, A(L))]}{\prod_{v \in M_K}[H^1(G_{v_L}, A(L_{v_L}))]},$$

where v_L is the fixed place of L lying above v for each $v \in M_K$.

Proof: It is obvious from Theorem 6 and Lemma 13. ∎

Because $H^1(G_{v_L}, A(L_{v_L})) = 0$ except for a finite number of places, the infinite product $\prod_{v \in M_K}[H^1(G_{v_L}, A(L_{v_L}))]$ is well-defined (see [5, Lemma 2.3]). Note that in the above theorem both sides are a power of 2.

In this study we assume that L/K is a finite Galois extension of number fields but we limit L/K to a quadratic extension in the latter half of section 4.

2. Tate–Shafarevich groups over Galois extensions

Denote the restriction map in the *Inflation–Restriction* sequence by res_A: $H^1(K, A) \to H^1(L, A)^G$. We have a natural commutative diagram (see [14, p. 296 and p. 335]):

(1)

$$\begin{array}{ccccccc} 0 \to H^1(G, A(L)) & \longrightarrow & H^1(K, A) & \xrightarrow{\mathrm{res}_A} & \mathrm{res}_A(H^1(K, A)) \to 0 \\ \downarrow \mathcal{F} & & \downarrow \mathcal{G} & & \downarrow \mathcal{H} \\ 0 \to \bigoplus_{v \in M_K} H^1(G_{v_L}, A(L_{v_L})) & \longrightarrow & \bigoplus_{v \in M_K} H^1(K_v, A) & \longrightarrow & \bigoplus_{v \in M_K} H^1(L_{v_L}, A), \end{array}$$

where v_L is the fixed place of L lying above v for each $v \in M_K$.

Let trans: $H^1(L,A)^G \to H^2(G, A(L))$ be the **Transgression** map (see section 4 for the definition). Denote by $\mathcal{I}$ the map $\mathrm{Coker}(\mathcal{F}) \to \mathrm{Coker}(\mathcal{G})$ induced from the above diagram. From [6, Theorem 2] we get $\mathrm{Ker(trans)} = \mathrm{res}_A(H^1(K,A))$. Therefore,

$$\mathrm{Ker}(\mathcal{H}) = Ш(A/L)^G \cap \mathrm{res}_A(H^1(K,A)) = Ш(A/L)^G \cap \mathrm{Ker(trans)}.$$

Note that $\mathrm{Ker}(\mathcal{G}) = Ш(A/K)$. Then the *Kernel-Cokernel* sequence of diagram (1) becomes the following sequence:

$$\begin{aligned} 0 \longrightarrow \mathrm{Ker}(\mathcal{F}) \longrightarrow Ш(A/K) \longrightarrow Ш(A/L)^G \cap \mathrm{Ker(trans)} \\ \longrightarrow \mathrm{Coker}(\mathcal{F}) \longrightarrow \mathcal{I}(\mathrm{Coker}(\mathcal{F})) \longrightarrow 0. \end{aligned} \tag{2}$$

For a topological abelian group M, let $\widehat{M}$ be the completion of M with respect to the topology defined by the subgroups of finite index. Write M^* for the group of continuous characters of finite order of M, i.e. $M^* = \mathrm{Hom}_{\mathrm{cts}}(M, \mathbf{Q}/\mathbf{Z})$.

THEOREM 1 (Global Duality Theorem): *Assume that $Ш(A/K)$ is finite. Then there is an exact sequence:*

$$0 \to Ш(A/K) \to H^1(K,A) \to \bigoplus_{v \in M_K} H^1(K_v, A) \to \widehat{A'(K)}^* \to 0,$$

where A' is the dual variety of A.

Proof: See [1, Corollary 1], [3, Theorem 1.1] or [8, I.6.14(b)]. ∎

THEOREM 2 (Local Duality Theorem): *For a place $v \in M_K$ there exists a bilinear, non-degenerate pairing*

$$\langle\ ,\ \rangle\colon H^0(K_v, A') \times H^1(K_v, A) \longrightarrow \mathbf{Q}/\mathbf{Z}.$$

Proof: See [16, p. 156-04], [17, p. 289] and [8, I.3.4 and I.3.7]. ∎

Here $H^0(K_v, A') = A'(K_v)$ unless v is archimedian, in which case it equals the quotient of $A'(K_v)$ by its identity component (see [17, p. 289]).

LEMMA 3: *The dual of the exact sequence*

$$0 \to H^1(G_{v_L}, A(L_{v_L})) \to H^1(K_v, A) \to H^1(L_{v_L}, A)$$

is the exact sequence

$$0 \leftarrow \widehat{H}^0(G_{v_L}, A'(L_{v_L})) \leftarrow H^0(K_v, A') \xleftarrow{N} H^0(L_{v_L}, A'),$$

where the map N is the norm map (the map tr *in [16, p. 156-04]).*

Proof: It is obvious from the local duality theorem and [16, (12) on p. 156-04]. For the archimedian primes, see [8, I.3.7]. ∎

LEMMA 4: *Suppose that M is a finite abelian group and that M' is an abelian group. Let $f\colon M \to M'$ be a group homomorphism and let $\mathrm{Hom}(f,\cdot)\colon \mathrm{Hom}(M',\mathbf{Q}/\mathbf{Z}) \to \mathrm{Hom}(M,\mathbf{Q}/\mathbf{Z})$ be the dual of f. Then [image of $\mathrm{Hom}(f,\cdot)$] = [image of f].*

Proof: It is obvious. ∎

LEMMA 5: *Let $\mathcal{F}_0'\colon \widehat{H}^0(G,A'(L)) \to \prod_{v\in M_K} \widehat{H}^0(G_{v_L},A'(L_{v_L}))$. Then*

$$[\mathcal{I}(\mathrm{Coker}(\mathcal{F}))] = [\widehat{H}^0(G,A'(L))/\mathrm{Ker}(\mathcal{F}_0')].$$

Proof: From diagram (1) there is the following commutative diagram:

$$\text{(3)}\qquad \begin{array}{ccc} \bigoplus_{v\in M_K} H^1(G_{v_L},A(L_{v_L})) & \longrightarrow & \bigoplus_{v\in M_K} H^1(K_v,A) \\ {\scriptstyle\text{surjective}}\big\downarrow & & \big\downarrow \\ \mathrm{Coker}(\mathcal{F}) & \xrightarrow{\quad\mathcal{I}\quad} & \mathrm{Coker}(\mathcal{G}). \end{array}$$

From Lemma 3 and [8, I.6.14(b)], the dual of a composition map in the above diagram,

$$\text{(4)}\qquad \bigoplus_{v\in M_K} H^1(G_{v_L},A(L_{v_L})) \to \bigoplus_{v\in M_K} H^1(K_v,A) \to \mathrm{Coker}(\mathcal{G}),$$

is the composition map

$$\text{(5)}\qquad \prod_{v\in M_K} \widehat{H}^0(G_{v_L},A'(L_{v_L})) \leftarrow \prod_{v\in M_K} H^0(K_v,A') \leftarrow \widehat{A'(K)}.$$

Now diagram (3) implies $[\mathcal{I}(\mathrm{Coker}(\mathcal{F}))]$ = [image of the map (4)] and Lemma 4 implies [image of the map (4)] = [image of the map (5)]. From the following natural commutative diagram:

$$\begin{array}{ccc} \prod_{v\in M_K} \widehat{H}^0(G_{v_L},A'(L_{v_L})) & \longleftarrow & \prod_{v\in M_K} H^0(K_v,A') \\ {\scriptstyle\mathcal{F}_0'}\big\uparrow & & \big\uparrow \\ \widehat{H}^0(G,A'(L)) & \xleftarrow{\quad\text{surjective}\quad} & \widehat{A'(K)}, \end{array}$$

[image of the map (5)] = [image of $\mathcal{F}_0'$] = $[\widehat{H}^0(G, A'(L))/\operatorname{Ker}(\mathcal{F}_0')]$. Then the lemma follows. ∎

THEOREM 6: *Assume that* $\text{Ш}(A/L)$ *is finite. Then*

$$\frac{[\text{Ш}(A/L)^G]}{[\text{Ш}(A/K)]} = \frac{[\operatorname{trans}(\text{Ш}(A/L)^G)][\operatorname{Ker}(\mathcal{F}_0')]}{[\widehat{H}^0(G, A'(L))][H^1(G, A(L))]} \prod_{v \in M_K} [H^1(G_{v_L}, A(L_{v_L}))].$$

Proof: From the map $\mathcal{F}$ in diagram (1), we have

$$\frac{[\operatorname{Coker}(\mathcal{F})]}{[\operatorname{Ker}(\mathcal{F})]} = \frac{[\bigoplus_v H^1(G_{v_L}, A(L_{v_L}))]}{[H^1(G, A(L))]}.$$

Then from the sequence (2) and Lemma 5, the theorem is immediate. ∎

COROLLARY 7 (Generalization of Main Theorem in [5]): *Suppose that* $\widehat{H}^0(G, A'(L)) = H^2(G, A(L)) = 0$. *Then*

$$\frac{[\text{Ш}(A/L)^G]}{[\text{Ш}(A/K)]} = \frac{\prod_{v \in M_K} [H^1(G_{v_L}, A(L_{v_L}))]}{[H^1(G, A(L))]}.$$

Proof: It is obvious from the previous theorem because $\operatorname{Ker}(\mathcal{F}_0') \subset \widehat{H}^0(G, A'(L))$ and because $\operatorname{trans}(\text{Ш}(A/L)^G) \subset H^2(G, A(L))$. ∎

3. Cassels pairing

When $\text{Ш}(A/K)$ is finite, there is a canonical pairing

$$\text{Ш}(A/K) \times \text{Ш}(A'/K) \longrightarrow \mathbf{Q}/\mathbf{Z},$$

which is non-degenerate. This pairing will be called Cassels pairing. For details, see [4], [17, p. 292] and [8, pp. 96–99]. The following is one definition of Cassels pairing in [8, pp. 96–97], which is called "The Weil pairing definition" in [10, 12.2].

For an abelian group M, let M_m denote the kernel of multiplication by m on M with an integer m. Pick a positive integer m which is a multiple of $[\text{Ш}(A/K)]$. Let $a \in \text{Ш}(A/K)$ and $a' \in \text{Ш}(A'/K)$. Choose elements b and b' of $H^1(K, A_m)$ and $H^1(K, A_m')$ mapping to a and a' respectively. For each $v \in M_K$, a maps to zero in $H^1(K_v, A)$, and from the diagram

$$\begin{array}{ccccc} A(K_v) & \longrightarrow & H^1(K_v, A_m) & \longrightarrow & H^1(K_v, A) \\ \| & & \uparrow & & \\ A(K_v) & \longrightarrow & H^1(K_v, A_{m^2}) & & \end{array}$$

we can lift b_v to an element $b_{v,1} \in H^1(K_v, A_{m^2})$ that is in the image of $A(K_v)$. Let β be a cocycle representing b, and choose a cochain $\beta_1 \in C^1(K, A_{m^2})$ such that $m\beta_1 = \beta$. Choose a cocycle $\beta_{v,1} \in Z^1(K_v, A_{m^2})$ representing $b_{v,1}$, and a cocycle $\beta' \in Z^1(K, A'_m)$ representing b'. The coboundary $d\beta_1$ of β_1 takes values in A_m, and $d\beta_1 \cup \beta'$ represents an element of $H^3(K, \overline{K}^\times) = 0$. So $d\beta_1 \cup \beta' = d\epsilon$ for some 2-cochain $\epsilon \in C^2(K, \overline{K}^\times)$. Now $(\beta_{1,v} - \beta_{v,1}) \cup \beta'_v - \epsilon_v$ is a 2-cocycle, and we define

$$\langle a, a' \rangle = \sum_{v \in M_K} \mathrm{inv}_v((\beta_{1,v} - \beta_{v,1}) \cup \beta'_v - \epsilon_v) \in \mathbf{Q}/\mathbf{Z}.$$

Remember that the cup-product is induced by the Weil pairing

$$e_m\colon A_m \times A'_m \to \overline{K}^\times,$$

and inv_v is the canonical map $H^2(K_v, \overline{K_v}^\times) \xrightarrow{\sim} \mathbf{Q}/\mathbf{Z}$.

Let $\langle -, - \rangle_K\colon Ш(A/K) \times Ш(A'/K) \to \mathbf{Q}/\mathbf{Z}$ be the Cassels pairing for A/K, and let $\langle -, - \rangle_L\colon Ш(A/L) \times Ш(A'/L) \to \mathbf{Q}/\mathbf{Z}$ be the Cassels pairing for A/L.

Write cores for the corestriction map $H^1(L, A) \to H^1(K, A)$ (for the definition see [11] or [15, p. 259]). Furthermore, cores and res_A can be defined on the cochain level and the transfer formula,

$$\mathrm{cores}(\mathrm{res}_A(u) \cup v) = u \cup \mathrm{cores}(v), \tag{6}$$

holds on the cochain level.

For details, see [2, III.9 and proof of V.3.8].

THEOREM 8: *For $a \in Ш(A/K)$ and $a' \in Ш(A'/L)$,*

$$\langle a, \mathrm{cores}(a') \rangle_K = \langle \mathrm{res}(a), a' \rangle_L.$$

Proof: Let m denote a positive common multiple of $[Ш(A/K)]$ and $[Ш(A'/L)]$. Given $a \in Ш(A/K)$ and $a' \in Ш(A'/L)$, by following the definition of Cassels pairing, we choose

$$\beta_1 \in C^1(K, A_{m^2}), \quad \beta_{v,1} \in Z^1(K_v, A_{m^2}) \quad \text{and} \quad \beta' \in Z^1(L, A'_m).$$

Then pick a 2-cochain $\epsilon \in C^2(L, L_s^\times)$ such that $d(\mathrm{res}_A(\beta_1)) \cup \beta' = d\epsilon$. When applying the map cores to this equality, the transfer formula (6) implies $d\beta_1 \cup \mathrm{cores}(\beta') = d\,\mathrm{cores}(\epsilon)$. For $w \in M_L$ lying over $v \in M_K$ write res_w, cores_w for the local restriction map $H^1(K_v, A) \to H^1(L_w, A)$, the local corestriction map $H^1(L_w, A) \to H^1(K_v, A)$, respectively.

Now let $c_v = (\beta_{1,v} - \beta_{v,1}) \cup \mathrm{cores}(\beta')_v - \mathrm{cores}(\epsilon)_v$ and let $d_w = (\mathrm{res}_A(\beta_1)_w - \mathrm{res}_w(\beta_{v,1})) \cup \beta'_w - \epsilon_w$. Then

$$\langle a, \mathrm{cores}(a')\rangle_K = \sum_{v\in M_K} \mathrm{inv}_v(c_v) \quad \text{and} \quad \langle \mathrm{res}(a), a'\rangle_L = \sum_{w\in M_L} \mathrm{inv}_w(d_w).$$

For $w \in M_L$ above $v \in M_K$, it is obvious that $\mathrm{res}_A(\beta_1)_w = \mathrm{res}_w(\beta_{1,v})$, and from [12, Lemma 2], $\sum_{w|v} \mathrm{cores}_w(\beta'_w) = \mathrm{cores}(\beta')_v$ and $\sum_{w|v} \mathrm{cores}_w(\epsilon_w) = \mathrm{cores}(\epsilon)_v$. Then the transfer formula (6) implies $\sum_{w|v} \mathrm{cores}_w(d_w) = c_v$. Therefore, from [13, p. 167 Proposition 1 ii)] we have

$$\sum_{w|v} \mathrm{inv}_w(d_w) = \sum_{w|v} \mathrm{inv}_v(\mathrm{cores}_w(d_w)) = \mathrm{inv}_v(c_v).$$

Then the theorem follows. ∎

COROLLARY 9: *We have the isomorphism*

$$\mathrm{Ker}(\mathrm{res}_A) \cap Ш(A/K) \cong \mathrm{Hom}(Ш(A'/K)/\,\mathrm{cores}(Ш(A'/L)), \mathbf{Q}/\mathbf{Z}).$$

Proof: From the previous theorem, it is obvious. ∎

LEMMA 10: *We write ${}_NШ(A/L)$ for the kernel of the norm map $N\colon Ш(A/L) \to Ш(A/L)^G$. Let $\mathrm{res}_{A'}: H^1(K, A') \to H^1(L, A')^G$ be the restriction map. Then*

$$\frac{[Ш(A/K)]}{[N(Ш(A/L))]} = [\mathrm{cores}({}_NШ(A/L))][\mathrm{Ker}(\mathrm{res}_{A'}) \cap Ш(A'/K)].$$

Proof: Note that $\mathrm{res} \circ \mathrm{cores} = N$ (see [2, III.9.5(iii)]). Then because

$$\mathrm{cores}({}_NШ(A/L)) = \mathrm{Ker}(\mathrm{res}) \cap \mathrm{cores}(Ш(A/L)),$$

we have an exact sequence

$$0 \to \mathrm{cores}({}_NШ(A/L)) \to \mathrm{cores}(Ш(A/L)) \xrightarrow{\mathrm{res}} N(Ш(A/L)) \to 0.$$

Then the lemma is immediate from the previous corollary. ∎

4. Transgression and Corestriction

Note that each element in $H^2(G, A(L))$ is represented by a normalized 2-cocycle $Y \in Z^2(G, A(L))$, that is, $Y(\tau, 1) = Y(1, \tau) = 0$, for $\tau \in G$.

Here we will introduce one definition of the **Transgression** map $\mathrm{trans}\colon H^1(L, A)^G \to H^2(G, A(L))$. For an element $x \in H^1(L, A)^G$, $\mathrm{trans}(x)$

is defined by the following condition: there are a cochain $f \in C^1(K, A)$ and a normalized 2-cocycle $Y \in Z^2(G, A(L))$ such that $f|_{G_L}$, the restriction of f to G_L, is a cocycle in $Z^1(L, A)$ representing x, the coboundary df of f is the natural image in $Z^2(K, A)$ of Y by inflation, and trans$(x) \in H^2(G, A(L))$ is the element determined by Y, that is,

$$\begin{array}{ccc} x \in H^1(L,A)^G & \xleftarrow{\text{res}} & f \in C^1(K,A) \\ {\scriptstyle\text{trans}}\downarrow & & \downarrow \\ Y \in Z^2(G,A(L)) & \xrightarrow{\text{inf}} & df \in Z^2(K,A). \end{array}$$

For more detail, see [6, p. 129]. Then for $\tau_1, \tau_2 \in G_K$,

$$Y(\widetilde{\tau_1}, \widetilde{\tau_2}) = df(\tau_1, \tau_2) = -f(\tau_1\tau_2) + f(\tau_1) + \tau_1 f(\tau_2), \tag{7}$$

where $\widetilde{\tau}$ means the coset τG_L.

From now on we assume that L is a quadratic extension of K. Here is the definition of the **Corestriction** map cores. For a cocyle $X \in Z^1(L, A)$ define cores$(X) \in Z^1(K, A)$ by

$$\text{cores}(X)(\tau) = \begin{cases} X(\tau) + \sigma X(\sigma^{-1}\tau\sigma) & \text{for } \tau \in G_L \\ X(\tau\sigma) + \sigma X(\sigma^{-1}\tau) & \text{for } \tau \in G_K - G_L \end{cases}$$

with fixed $\sigma \in G_K - G_L$. See [15, p. 259] or [11, p. 77 and Theorem 3].

Let χ denote the non-trivial character of G. Write A^χ for the twist of A by χ (see [7, §2]). Then there is an isomorphism ϕ: $A \to A^\chi$ defined over L such that $\widetilde{\tau}(\phi) = \chi(\widetilde{\tau})\phi = -\phi$ for $\tau \in G_K - G_L$.

LEMMA 11: *Define* cores $\circ H^1(\cdot, \phi)$: $H^1(L, A)^G \to H^1(K, A^\chi)$ *by the composition of the following two maps:*

$$H^1(L,A)^G \xrightarrow{H^1(\cdot,\phi)} H^1(L,A^\chi) \xrightarrow{\text{cores}} H^1(K,A^\chi).$$

Then Ker(cores $\circ H^1(\cdot, \phi)$) = Ker(trans).

Proof: If a cochain $f \in C^1(K, A)$ and a normalized 2-cocycle $Y \in Z^2(G, A(L))$ satisfy (7), then by direct computation it is obvious that

$$(\text{cores} \circ H^1(\cdot, \phi))(f|_{G_L})(\tau)$$

is equal to

$$\begin{cases} (1-\tau)(\phi(f(\sigma))) & \text{if } \tau \in G_L, \\ (1-\tau)(\phi(f(\sigma))) - \phi(Y(\widetilde{\sigma}, \widetilde{\sigma})) & \text{if } \tau \in G_K - G_L, \end{cases}$$

with the fixed $\sigma \in G_K - G_L$.

When $f|_{G_L} \in \mathrm{Ker}(\mathrm{trans})$, that is, $Y(\tilde{\sigma},\tilde{\sigma}) = 0$, it is obvious that $f|_{G_L} \in \mathrm{Ker}(\mathrm{cores} \circ H^1(\cdot,\phi))$.

Now suppose $f|_{G_L} \in \mathrm{Ker}(\mathrm{cores} \circ H^1(\cdot,\phi))$. Then there is $Q \in A^\chi(L)$ such that $\phi(Y(\tilde{\sigma},\tilde{\sigma})) = Q - \sigma(Q)$. So $Y(\tilde{\sigma},\tilde{\sigma}) = \phi^{-1}(Q) + \sigma(\phi^{-1}(Q))$. Define $g \in C^1(G, A(L))$ by $g(1) = 0$ and $g(\tilde{\sigma}) = \phi^{-1}(Q)$. Then $df = Y = dg$. Therefore, $f|_{G_L} = (f-g)|_{G_L}$ and $d(f-g) = 0$. So $f - g \in Z^1(K, A)$. Therefore, $f|_{G_L} = (f-g)|_{G_L} \in \mathrm{Ker}(\mathrm{trans})$. ∎

Corollary 12: *Suppose that $Ш(A/L)^G$ is finite. Then*

$$[\mathrm{trans}(Ш(A/L)^G)] = [\mathrm{cores}({}_NШ(A^\chi/L))].$$

Proof: Note that the previous lemma implies

$$[\mathrm{trans}(Ш(A/L)^G)] = [(\mathrm{cores} \circ H^1(\cdot,\phi))(Ш(A/L)^G)].$$

Note that $H^1(\cdot,\phi)$ is injective and $H^1(\cdot,\phi)(Ш(A/L)^G) =_N Ш(A^\chi/L)$ (see (2) of [5]). So the corollary follows. ∎

Lemma 13: *Assume $Ш(A^\chi/K)$ is finite. Then*

$$\frac{[Ш(A^\chi/K)]}{[(1-\sigma)Ш(A/L)]} = [\mathrm{trans}(Ш(A/L)^G)][\mathrm{Ker}(\mathcal{F}_0')].$$

Proof: From Lemma 10, we get

$$\frac{[Ш(A^\chi/K)]}{[N(Ш(A^\chi/L))]} = [\mathrm{cores}({}_NШ(A^\chi/L))][\mathrm{Ker}(\mathrm{res}_{A^{\chi'}}) \cap Ш(A^{\chi'}/K)].$$

Note that $N(Ш(A^\chi/L)) \cong (1-\sigma)Ш(A/L)$ through ϕ defined before Lemma 11. From the following diagram

$$\begin{array}{ccc} H^1(G, A^{\chi'}(L)) & \longrightarrow & \bigoplus_v H^1(G_{v_L}, A^{\chi'}(L_{v_L})) \\ \cong\uparrow & & \cong\uparrow \\ \widehat{H}^0(G, A'(L)) & \xrightarrow{\mathcal{F}_0'} & \bigoplus_v \widehat{H}^0(G_{v_L}, A'(L_{v_L})) \end{array}$$

we know that $\mathrm{Ker}(\mathcal{F}_0')$ is isomorphic to the kernel of the upper horizontal map, which is equal to $\mathrm{Ker}(\mathrm{res}_{A^{\chi'}}) \cap Ш(A^{\chi'}/K)$ (see diagram (1)). Note that the vertical isomorphisms are induced from the isomorphism defined over L between A' and its quadratic twist $A^{\chi'}$. Then the lemma is immediate from the previous corollary. ∎

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